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An approach to the many-exciton system

Ba An Nguyen and Ngoc Cam Hoang
Institute of Theoretical Physics, Academy of Sciences of Vietnam, PO Box 429 Bo Ho, Hanoi 10000, Vietnam

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Abstract. A rather direct way of studying a high-density exciton system is outlined to take full account of the non-boson behaviour of excitons. Many-exciton systems are shown to be always non-ideal and thus they can only be characterised by their total energy and total number. The single-exciton energy and distribution function can be understood simply in the framework of the mean-field approximation and they both depend upon the exciton density. It is shown that our approach could be applied to both Frenkel and Wannier–Mott excitons in materials of bulk and low-dimensional structures as well as under the action of an external field.

1. Introduction

Despite the fact that the concept of excitons was introduced nearly 60 years ago (Frenkel 1931, Wannier 1937, Mott 1938) their collective properties have been seriously studied only for the last two decades since the original paper by Keldysh (1968).

Nowadays using short-pulse technique one may generate a high electron–hole density in a semiconductor. At very high excitation or on a very short time scale the carrier distributions turn out to differ radically from the thermal distributions and the theoretical tool that seems most adapted to these circumstances is the non-equilibrium Green function method first developed by Keldysh (1965) and then widely used, for example, by Haug (1985), Henneberger (1986), May (1986), Schafer and Treusch (1986) and Glaeske and Schubert (1988). At moderate excitation below the Mott transition the electron–hole attractive Coulomb potential is still strong enough to form bound pair states called excitons. The temporal evolution of the exciton system is governed by two kinds of time. The first kind $\tau_\perp$—the transverse or dephase time—characterises the rate of exciton–exciton scattering. The second kind $\tau_\parallel$—the longitudinal or depopulation time—characterises the rate of electron–hole recombinations and hence limits the lifetime of excitons. If $\tau_\parallel \ll \tau_\perp$, excitons recombine rather than interact with each other. In the reverse limit ($\tau_\parallel \gg \tau_\perp$) they have enough time to acquire quasi-equilibrium distributions among themselves and the exciton system can be characterised by a certain density n and an effective temperature T. The latter, of course, must be assumed to be as low as $k_B T < I^b$ (k_B and I^b are the Boltzmann constant and exciton binding energy, respectively) to prevent excitons from thermal ionisations. Since in typical semiconductors $\tau_\parallel$ and $\tau_\perp$ are of the order of some nanoseconds and picoseconds, respectively, one really has, under resonant excitation in the exciton spectral region, an exciton gas

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which can reach a thermal quasi-equilibrium within a duration τ as long as $\tau_\perp < \tau < \tau_\parallel$. For such a gas, many-exciton effects play a very important role. They cause many new phenomena that are absent at low excitation, e.g. the formation of biexcitons, the occurrence of a phase transition or condensations in real or reciprocal spaces, and the appearance of multi-stability and instability.

So far two theoretical descriptions of dense exciton systems in semiconductors have been given, namely the pairing theory (Keldysh and Kozlov 1968) and the boson formalism (Hanamura 1970, 1974, Steyn-Ross and Gardiner 1983). In spite of their success in spectral problems many-exciton systems do not seem to be so well understood, especially from the statistical point of view. Kaplan (1976) formulated specific statistics of Frenkel excitons which then have been fruitfully utilised for researching the Bose–Einstein condensation (Kaplan and Ruvinskii 1976), the excitation-induced change of giant polariton dispersions (Avdjugin *et al* 1983; Nguyen 1988b, 1989a, b) and the density and optical bi-stability (Nguyen 1988a) of excitons in molecular media.

In this paper we would like to extend Kaplan's results to Wannier–Mott excitons. Now, instead of exponent functions in unitary transformations connecting exciton operators in coordinate and momentum spaces (Kaplan 1976), we have to handle hydrogen-like functions describing the relative electron–hole motion in an exciton. This makes the situation quite confused and one hardly expects to obtain final results in a compact form! Fortunately, we can show that our results are generalised and they may be used for any kinds of exciton provided that their bound functions form a complete and orthonormalised set. Comments on Kaplan's formulations as well as the advantage of our approach in comparison with the pairing theory and the boson representation will be given in due course. Everywhere in this paper we work in the $\hbar = c = 1$ unit system.

2. Exciton operators and their commutators

For simplicity, we restrict ourselves to a direct two-band semiconductor with isotropic effective masses m_e and m_h and disregard the carrier spin.

The pairing theory (Keldysh and Kozlov 1968) starts with the following Hamiltonian for the electron–hole system:

$$H = \sum_k \Big(E_e(k) e_k^+ e_k + E_h(k) h_k^+ h_k + \tfrac{1}{2} \sum_{pq} U_q [e_{k+q}^+ e_{p-q}^+ e_p e_k + h_{k+q}^+ h_{p-q}^+ h_p h_k - 2 e_{k+q}^+ h_{p-q}^+ h_p e_k] \Big) \tag{1}$$

where e_k (h_k) destroys an electron (a hole) with momentum k and energy $E_e(k)$ ($E_h(k)$). $U_q = 4\pi e^2/q^2 V$, where V is the normalised volume of the sample. At very low excitation the electron–hole bound pair eigenvalue E and eigenfunction $|X\rangle$ and H can be found from the solution of the equation $H|X\rangle = E|X\rangle$, yielding

$$E \equiv E_{\nu k} = E_g - I_\nu^b + k^2/2(m_e + m_h) \tag{2}$$

$$|X\rangle = b_{\nu k}^+ |0\rangle \qquad b_{\nu k}^+ = V^{-1/2} \sum_p f_\nu(p - \beta k) e_{k-p}^+ h_p^+ |0\rangle \tag{3}$$

where ν specifies the orbital state of the electron–hole relative motion in an exciton,

$\beta = m_h/(m_e + m_h)$ and $|0\rangle$ is the semiconductor vacuum state. The hydrogen-like wavefunctions f_ν satisfy the ortho-normalisation and completeness conditions

$$\sum_p f_\nu^*(p) f_{\nu'}(p) = V\delta_{\nu\nu'} \tag{4}$$

$$\sum_\nu f_\nu^*(p) f_\nu(p') = V\delta_{pp'}. \tag{5}$$

For operators $b_{\nu k}$ and $b_{\nu k}^+$ referred to as exciton operators the following commutators hold:

$$[b_{\nu k}, b_{\nu' k'}] = [b_{\nu k}^+, b_{\nu' k'}^+] = 0 \tag{6}$$

$$[b_{\nu k}, b_{\nu' k'}^+] = \delta_{\nu\nu'}\delta_{kk'} - V^{-1}\sum_p f_\nu^*(p - \beta k)[f_{\nu'}(p - \beta k')e_{k'-p}^+ e_{k-p} + f_{\nu'}(p - k + \alpha k')h_{p+k'-k}^+ h_p] \tag{7}$$

where $\alpha = 1 - \beta$. Note that statements such as '$b_{\nu k}$ destroys an exciton in the state ν with energy $E_{\nu k}$' are true only in the pair vanishing limit. Later we shall clarify the physical meaning of exciton operators in the case of dense exciton systems.

As to the boson formalism, Hanamura (1970) uses a transformation of Usui (1960) to map the fermion space onto a hypothetical boson space in which excitons behave as ideal bosons and their mutual interactions should be described in an effective manner. However, the transformation due to Usui does not ensure a one-to-one correspondence between fermion and boson states and the transformed Hamiltonian is non-Hermitian. To overcome these inconsistencies, Hanamura (1974) has to resort to an arranging procedure which in our opinion is quite complicated and not natural. To avoid the above-mentioned difficulty, Steyn-Ross and Gardiner (1983) apply a modified version of the transformation by Marumori *et al* (1964) and have changed H to H^B expressed through the ideal boson operators $c_{\nu k}$ and $c_{\nu k}^+$ as follows:

$$H^B = \sum_{\nu k} E_{\nu k} c_{\nu k}^+ c_{\nu k} + \tfrac{1}{2}\sum_{\substack{kpq \\ \nu\nu'\nu''\nu'''}} W_{\nu\nu'\nu''\nu'''}(kpq) c_{\nu k+q}^+ c_{\nu' p-q}^+ c_{\nu'' p} c_{\nu''' k} \tag{8}$$

where the effective boson–boson coupling functions W comprise terms of two kinds: one represents the dynamic interaction among bosons and the other comes from the dynamic–kinematic interboson interaction which partly reflects the non-boson character of excitons but, as one can see, the boson formalism is unable to cover all properties of the very complicated system of real excitons, i.e. it omits the so-called purely kinematic interaction (see, e.g., Lalovic *et al* 1969). To take full account of the non-boson nature, we formally follow the procedure performed in the paper by Steyn-Ross and Gardiner (1983) but we replace in it the boson operators $c_{\nu k}$ and $c_{\nu k}^+$ by the exciton operators $b_{\nu k}$ and $b_{\nu k}^+$. Further, to highlight the role of the purely kinematic interaction, we write the transformed Hamiltonian labelled $\tilde{H}$, neglecting the interaction of dynamic and dynamic–kinematic kinds:

$$\tilde{H} = \sum_{\nu k} E_{\nu k} b_{\nu k}^+ b_{\nu k}. \tag{9}$$

Since $b_{\nu k}$ and $b_{\nu k}^+$ are not bosonic, let us conventionally call our transformation procedure excitonisation and not bosonisation. The commutation relations (6) and (7) are not

closed because they involve simultaneously both exciton and electron (hole) operators. Following Kaplan (1976) and owing to the reverse-to-(3) relations

$$e_q^+ h_p^+ = V^{-1/2} \sum_\nu f_\nu^*(\alpha p - \beta q) b_{\nu p+q}^+ \tag{10}$$

we construct the trilinear commutators for $b_{\nu k}$ and $b_{\nu k}^+$ such as

$$[[b_{\nu k}, b_{\nu' k'}^+], b_{\nu'' k''}] = \sum_{\nu'''} A_{kkk''k''}^{\nu\nu'\nu''\nu'''} b_{\nu''' k-k'+k''} \tag{11}$$

$$[[b_{\nu k}, b_{\nu' k'}^+], b_{\nu'' k''}^+] = -\sum_{\nu'''} A_{kk',k''+k'-k,k''}^{\nu\nu'\nu'''\nu''} b_{\nu''' k''+k'-k}^+ \tag{12}$$

$$A_{kk'k''k'''}^{\nu\nu'\nu''\nu'''} = V^{-2} \sum_p [f_\nu^*(p-\beta k) f_{\nu'}(p-\beta k') f_{\nu''}^*(p-k'+\alpha k'') f_{\nu'''}(p-k+\alpha k''')$$
$$+ f_{\nu'''}(p-\beta k''') f_\nu^*(p-\beta k) f_{\nu'}(p-k+\alpha k') f_{\nu''}^*(p-k'''+\alpha k'')]. \tag{13}$$

Equations (6) and (11)–(13) form a closed set of exact commutation relations for non-boson exciton operators which resemble those for parafermions (Kalnay 1975, Kalnay and MacContrina 1976). Owing to the hydrogen-like functions, equations (11)–(13) look much more cumbrous than those for Frenkel excitons (Kaplan 1976).

3. Exciton total number operator

At first view, $\hat{H}$ represents non-interacting excitons. Yet, it is so only if $b_{\nu k}$ and $b_{\nu k}^+$ were bosonic. The deviation of $[b_{\nu k}, b_{\nu' k'}^+]$ from Kronecker symbols and the omission of them in $[[b_{\nu k}, b_{\nu' k'}^+], b_{\nu'' k''}]$, $[[b_{\nu k}, b_{\nu' k'}^+], b_{\nu'' k''}^+]$ tell us that excitons cannot be truly free as elementary particles in the quantum field theory (see, e.g., Akhiezer and Berestexkii 1969, Nguyen 1984). We shall prove this later but even now we make use of it to be aware that there exist no stationary states containing a certain number of excitons with given values of ν and k; so the many-exciton system should be characterised by its total number, whose operator is estimated differently from that of fermions and bosons. Define the following operator:

$$\hat{N} = \tfrac{1}{2} \mathfrak{N}^{-1} \sum_{\nu k} ([b_{\nu k}^+, b_{\nu k}] + 1) \tag{14}$$

where $\mathfrak{N}$ is the number of unit cells which, as is well known, is equal to the number of states $\Sigma_k 1$ in a band inside the first Brillouin zone. Using (4) and (5) we can also check that the number of states and the number of orbitals are equal, i.e. (see appendix)

$$\mathfrak{N} = \sum_k 1 = \sum_\nu 1. \tag{15}$$

With the aid of (4)–(7) and (11)–(15) we can derive the following relations valid for any ν, k and constant γ:

$$\hat{N} = \tfrac{1}{2} \sum_p (e_p^+ e_p + h_p^+ h_p) \tag{16}$$

$$[\hat{N}, \gamma b_{\nu k}] = -\gamma b_{\nu k} \qquad [\hat{N}, \gamma b_{\nu k}^+] = \gamma b_{\nu k}^+. \tag{17}$$

Denote by $|N\rangle$ the eigenfunction of $\hat{N}$ corresponding to the eigen-value N:

$$\hat{N}|N\rangle = N|N\rangle. \tag{18}$$

The second-quantisation method presented, for example, in a book by Nguyen (1984), the use of (16) and (17) and the fact that e and h are fermion operators given by

$$\prod_{i=1}^{\mathfrak{N}+1} e_{p_i} = \prod_{i=1}^{\mathfrak{N}+1} h_{p_i} = \prod_{i=1}^{\mathfrak{N}+1} e_{p_i}^{+} = \prod_{i=1}^{\mathfrak{N}+1} h_{p_i}^{+} = 0 \tag{19}$$

help us to prove that

$$N = 0, 1, 2, \ldots, \mathfrak{N} \tag{20}$$

$$b_{\nu k}|0\rangle = 0 \tag{21}$$

$$b_{\nu k}|N\rangle = u_N^{\nu k}|N-1\rangle \tag{22}$$

$$b_{\nu k}^{+}|N\rangle = v_{N+1}^{\nu k}|N+1\rangle \tag{23}$$

$$b_{\nu k}^{+}|\mathfrak{N}\rangle = 0 \tag{24}$$

where $u_N^{\nu k}$ and $v_{N+1}^{\nu k}$ are coefficients to be determined. Equations (20)–(24) together with equations (11)–(13) let us conclude that $\hat{N}$ is the exciton total number operator and the excitons could be treated as quasi-parafermions obeying rank-$\mathfrak{N}$ parfermion-like statistics. The indices ν and k here play only a formal role and will be dropped in what follows in this section. In the next section we shall work in the mean-field approximation (MFA) within which the physical nature of $b_{\nu k}$ and $b_{\nu k}^{+}$ will be cleared up and then the indices ν and k become necessary and meaningful. If $b_{\nu k}$ and $b_{\nu k}^{+}$ were bosonic or fermionic, the determination of u_N and v_{N+1} is a rather simple task (see, e.g., Nguyen 1984). Since it is not so, let us begin from (18) which gives

$$N = \langle N|\hat{N}|N\rangle. \tag{25}$$

Putting (14) into (25) and then expanding, we obtain with the use of (15), (22) and (23) a very helpful relation valid for any N:

$$2N = \mathfrak{N}(v_N u_N - u_{N+1}v_{N+1} + 1). \tag{26}$$

Now writing (26) as a system of equations for various N, i.e.

$$2 \times 0 = \mathfrak{N}(v_0 u_0 - u_1 v_1 + 1) \tag{27}$$

$$2 \times 1 = \mathfrak{N}(v_1 u_1 - u_2 v_2 + 1) \tag{28}$$

$$\ldots$$

$$2(N-1) = \mathfrak{N}(v_{N-1}u_{N-1} - u_N v_N + 1) \tag{29}$$

and summing separately two sides of equations (27)–(29) with attention to the cancellation of terms in two sequential rows, we get

$$2[0 + 1 + 2 + \ldots + (N-1)] = \mathfrak{N}(v_0 u_0 - u_N v_N + N). \tag{30}$$

Taking into account $u_0 = 0$ (see (21) and (22)) and $u_N = v_N^{*}$ (see (22) and (23)) we

immediately from (30) determine u_N and v_N to the accuracy of an unimportant phase factor:

$$u_N = v_N = \{N[1 - (N-1)/\mathfrak{N}]\}^{1/2}. \tag{31}$$

Then equations (22) and (23) become

$$b|N\rangle = \{N[1 - (N-1)/\mathfrak{N}]\}^{1/2}|N-1\rangle \tag{32}$$

$$b^+|N\rangle = [(N+1)(1 - N/\mathfrak{N})]^{1/2}|N+1\rangle. \tag{33}$$

The functions $|N\rangle$, on the one hand, may be constructed by acting b^+ on $|0\rangle$ N times and, on the other hand, must be normalised to unity. Therefore, we can introduce the normalisation coefficient C_N as

$$|N\rangle = C_N \underbrace{b^+ \ldots b^+}_{N \text{ times}}|0\rangle. \tag{34}$$

Using (23), (31) and (33) we have

$$|N\rangle = C_N \underbrace{b^+ \ldots b^+}_{N-1} v_1|1\rangle = C_N \underbrace{b^+ \ldots b^+}_{N-2} v_1 v_2|2\rangle$$

$$= \ldots = C_N b^+ v_1 v_2 \ldots v_{N-1}|N-1\rangle = C_N v_1 v_2 \ldots v_{N-1} v_N|N\rangle. \tag{35}$$

The normalisation condition $\langle N|N\rangle = 1$ then gives

$$C_N = \left(\prod_{i=1}^{N} v_i\right)^{-1} = \left[N!\left(\frac{1-1}{\mathfrak{N}}\right)\left(\frac{1-2}{\mathfrak{N}}\right)\ldots\left(\frac{1-N-1}{\mathfrak{N}}\right)\right]^{-1/2}. \tag{36}$$

It is worth noting that the results of this section coincide with those of Kaplan (1976) in spite of the different methods of derivation and the fact that Kaplan considers Frenkel excitons while we are concerned with Wannier–Mott excitons. Because of the conditions (4) and (5) the hydrogen-like functions disappear totally in our calculations owing to summing over ν and k in (14). This interesting remark means that our results remain unchanged independent of the kind of exciton that we have to deal with. They may be excitons in an electric or magnetic field, in bulk or low-dimensional structure materials, etc., because, as a principle, their wavefunctions always satisfy conditions such as (4) and (5).

4. Density-dependent single-exciton energy

In this section we try to derive the expression for the single-exciton energy within an appropriate approximation called the MFA. In doing so, the operators $b_{\nu k}$ and $b^+_{\nu k}$ will gain their full physical meaning. From (9) and (14) it follows that

$$[\hat{H}, \hat{N}] = 0. \tag{37}$$

Equation (37) reveals that the total energy $\tilde{E}_N$ of the system is as important as its total

number N. Consequently, the functions $|N\rangle$ introduced in the previous section should be understood in the sense

$$|N\rangle \rightarrow \left|\begin{matrix}\tilde{E}_N \\ N\end{matrix}\right\rangle \equiv |N). \tag{38}$$

From (32) and (33) we already know that

$$b_{\nu k}\left|\begin{matrix}\tilde{E}_N \\ N\end{matrix}\right\rangle \sim \left|\begin{matrix}\tilde{E}_{N-1} \\ N-1\end{matrix}\right\rangle \tag{39}$$

$$b^{+}_{\nu k}\left|\begin{matrix}\tilde{E}_N \\ N\end{matrix}\right\rangle \sim \left|\begin{matrix}\tilde{E}_{N+1} \\ N+1\end{matrix}\right\rangle \tag{40}$$

and what $\tilde{E}_{N\pm 1}$ equals is the subject to be solved now. For this purpose, let us calculate the following matrix element:

$$(N-1|[\tilde{H}, b_{\nu k}]|N) = \sum_{\nu' k'} E_{\nu' k'}[(N-1|b^{+}_{\nu' k'} b_{\nu' k'} b_{\nu k}|N) - (N-1|b_{\nu k} b^{+}_{\nu' k'} b_{\nu' k'}|N)]. \tag{41}$$

In the MFA the matrix elements in (4) can be split off properly and the symbols $\delta_{\nu\nu'}\delta_{kk'}$ will appear to get rid of the sums. This procedure results in

$$(N-1|[\tilde{H}, b_{\nu k}]|N) \simeq -GE_{\nu k}(N-1|b_{\nu k}|N) \tag{42}$$

$$G = (N-1|[b^{+}_{\nu k}, b_{\nu k}]|N-1). \tag{43}$$

It can be judged that G is in fact independent of ν and k. Then we can use (14) and (25) to bring it into the form

$$G = 1 - (2/\mathfrak{N})(N-1|\hat{N}|N-1) = 1 - 2(N-1)/\mathfrak{N}. \tag{44}$$

Substituting (44) into (42) we get approximately

$$[\tilde{H}, b_{\nu k}] = -\varepsilon_{\nu k} b_{\nu k} \tag{45}$$

$$\varepsilon_{\nu k} = E_{\nu k}[1 - 2(N-1)/\mathfrak{N}]. \tag{46}$$

Similarly, we also have

$$[\tilde{H}, b^{+}_{\nu k}] = \varepsilon_{\nu k} b^{+}_{\nu k}. \tag{47}$$

Using (45) and (47) it is easy to prove (Nguyen 1984) that

$$b_{\nu k}\left|\begin{matrix}\tilde{E}_N \\ N\end{matrix}\right\rangle \sim \left|\begin{matrix}\tilde{E}_{N-1} \\ N-1\end{matrix}\right\rangle = \left|\begin{matrix}\tilde{E}_N - \varepsilon_{\nu k} \\ N-1\end{matrix}\right\rangle \tag{48}$$

$$b^{+}_{\nu k}\left|\begin{matrix}\tilde{E}_N \\ N\end{matrix}\right\rangle \sim \left|\begin{matrix}\tilde{E}_{N+1} \\ N+1\end{matrix}\right\rangle = \left|\begin{matrix}\tilde{E}_N + \varepsilon_{\nu k} \\ N+1\end{matrix}\right\rangle. \tag{49}$$

The relations (48) and (49) state that the action of $b_{\nu k}$ ($b^{+}_{\nu k}$) on a state $|N)$ takes from (adds to) it an exciton and at the same time decreases (increases) its total energy $\tilde{E}_N$ by an amount of $\varepsilon_{\nu k}$. Therefore, we can interpret $\varepsilon_{\nu k}$ as the energy of an exciton which as seen from (46) depends on the total number N of excitons of the system. As to $b_{\nu k}$ ($b^{+}_{\nu k}$) it serves as the annihilation (creation) operator of an exciton with energy $\varepsilon_{\nu k}$.

Owing to the above-stated N-dependence we may, when needed, write the index N as $b_{\nu k}$ ($b^+_{\nu k}$) and $\varepsilon_{\nu k}$ as $b_{\nu kN}$ ($b^+_{\nu kN}$) and $\varepsilon_{\nu kN}$.

Now it is timely to note that $\varepsilon_{\nu kN}|_{N=1} = E_{\nu k}$ and $\varepsilon_{\nu kN}|_{N>1} < E_{\nu k}$. These mean the existence of an attractive potential among excitons which is due simply to their non-boson character. So, such a potential is reasonably called purely kinematic and cannot be described by the boson formalism. Because of the purely kinematic interaction between excitons that is always present in the system with $N > 1$ an exciton gas is never ideal even when all the dynamic interactions are switched off.

5. Density-dependent exciton distribution function

After having understood the physical meaning of the operators $b_{\nu k}$ and $b^+_{\nu k}$ in the MFA sense we can define the exciton distribution function $n_{\nu kN}$ as usual:

$$n_{\nu kN} \equiv \langle b^+_{\nu kN} b_{\nu kN}\rangle \equiv (N-1|b^+_{\nu kN} b_{\nu kN}|N-1)$$
$$\equiv \mathrm{Sp}\{\exp[(1/k_B T)(\mu\hat{N} - \tilde{H})]\, b^+_{\nu kN} b_{\nu kN}\}/\mathrm{Sp}\{\exp[(1/k_B T)(\mu\hat{N} - \tilde{H})]\} \quad (50)$$

where μ is the chemical potential, $\hat{N}$ and $\tilde{H}$ being determined by (14) and (9), respectively. As the MFA applied to an N-particle system treats particles as independent in the sense that each of them moves in the averaged field produced by the $N-1$ particles, the averaging must be carried out over the state $|N-1)$ rather than $|N)$. This will lead to the physically correct vanishing density limit of the distribution function (see later). Making use of (37) and the invariance of $\mathrm{Sp}\{\ldots\}$ in regard to operator cyclic permutations we can write

$$\mathrm{Sp}\{\exp[(1/k_B T)(\mu\hat{N} - \tilde{H})]\, b^+_{\nu kN} b_{\nu kN}\} = \mathrm{Sp}\{\exp[(1/k_B T)(\mu\hat{N} - \tilde{H})]\, b_{\nu kN} B_{\nu kN}\} \quad (51)$$

where

$$B_{\nu kN} = \exp[(1/k_B T)(\mu\hat{N} - \tilde{H})]\, b^+_{\nu kN} \exp[-(1/k_B T)(\mu\hat{N} - \tilde{H})]. \quad (52)$$

At this moment, resorting to (17) and (47) we can put $B_{\nu kN}$ in the form

$$B_{\nu kN} = \exp[(1/k_B T)(\mu - \varepsilon_{\nu kN})]\, b^+_{\nu kN}. \quad (53)$$

Substituting (53) into (51), then (51) into (50) and applying equations (50) from the right to the left with account of (following from (14) and (15))

$$\langle bb^+\rangle = 1 + \langle b^+ b\rangle - (2/\mathfrak{N})\langle\hat{N}\rangle \quad (54)$$

we get an equation for $n_{\nu kN}$, whose solution reads

$$n_{\nu kN} = [1 - 2(N-1)/\mathfrak{N}]/(\exp[\![(1/k_B T)\{E_{\nu k}[1 - 2(N-1)/\mathfrak{N}]\}]\!] - 1). \quad (55)$$

Obviously, the exciton distribution function (55) is dependent on the exciton number N and the rank order $\mathfrak{N}$ of the parafermion-like statistics which they obey. Furthermore, in the low-density limit ($N \to 1$), (55) reproduces the well known Bose–Einstein function. Note also that the function derived by Kaplan and Ruvinskii (1976) did not reflect such a physically meaningful limit.

6. Conclusions

In conclusion, we prefer to discuss problems which still have to be dealt with rather than to summarise what has been done in this paper. A few of these remaining problems are as follows.

(i) The dynamic and dynamic–kinematic interactions must be added to $\tilde{H}$ on the same footing as in deriving $\tilde{H}$. Only in such a way will the relative contributions of the three kinds of interaction be accounted for reliably. It is expected that the purely kinematic interaction would give non-negligible contributions in special high-excitation phenomena. For an illustrative estimate let us take CdS with the parameters $E \equiv E_{1sk=0} = 2.5528\,\text{eV}$, $I \equiv I^{b}_{1s} = 32.9\,\text{meV}$, $r \equiv r_{1s}$ (exciton Bohr radius) $= 25.5\,\text{Å}$ and unit-cell size $a = 5.82\,\text{Å}$. The energy corrections scaled in the exciton density due to the purely kinematic and dynamic interactions then will be $|\Delta E_{\text{kin}}| \simeq 2EV/\mathfrak{N} = 2Ea^3 \simeq 1.0 \times 10^{-21}\,\text{eV cm}^3$ and $|\Delta E_{\text{dyn}}| \simeq 26\,\pi Ir^3/3 \simeq 1.5 \times 10^{-20}\,\text{eV cm}^3$ (see, e.g., Hanamura 1974). We see that $|\Delta E_{\text{kin}}|$ is about 7% of $|\Delta E_{\text{dyn}}|$, indicating the necessity of taking the former into account together with the latter.

(ii) The spin effect, the band structures, more adequate approximations, etc, should be considered if needed to explain concrete experimental findings.

(iii) Since the parafermions can be bosonised exactly without the loss of any of their properties (Kalnay 1975), the parafermion-like behaviour of the excitons suggests that we should bosonise the excitons in a way other than the existing methods. If it were successful, all kinds of interaction including the purely kinematic one could be described at the same time as considering excitons as ideal bosons.

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Appendix

From (4) and (5) it follows that, respectively,

$$\sum_{p} |f_{\nu}(p)|^2 = V \tag{A1}$$

$$\sum_{\nu} |f_{\nu}(p)|^2 = V. \tag{A2}$$

Summing both sides of (A1) over ν, and then using (A2), one has

$$\sum_{\nu}\left(\sum_{p}|f_{\nu}(p)|^2\right) = \sum_{p}\left(\sum_{\nu}|f_{\nu}(p)|^2\right) = V\sum_{p}1 = V\sum_{\nu}1 \tag{A3}$$

which gives the second equality of (15).

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